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Euclidean Space: Sequences in R^m Space and Important Sets in R^m Space

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ABSTRACT
Objective: *This article aims to provide comprehensive insights into the Euclidean space R^m, serving as a resource for students, educators, and researchers in mathematical analysis and linear algebra.* ***Method:*** *Fundamental concepts and properties of R^m are elaborated, including sequences, distance properties, and the characteristics of open and closed sets. The analysis incorporates mathematical frameworks to examine bounded and unbounded sets alongside hyperplanes within R^m.* ***Results:*** *The study elucidates the key mathematical constructs and properties that define R^m, offering a structured understanding of its theoretical and practical applications in mathematics.* ***Novelty:*** *This work stands out by consolidating essential definitions and properties of R^m in a single resource, bridging gaps between foundational and advanced topics in mathematical analysis.*

INTRODUCTION

Euclidean space R^m occupies a central position in various fields of mathematics, including analysis, linear algebra, and geometry [1], [2]. This space is a general concept where each element can be imagined as an m-dimensional vector. Euclidean space is significant not only for theoretical research but also for solving numerous practical problems [3].

This article focuses on the fundamental properties of sequences in R^m space and their important sets [4]. Initially, the basic concepts of Euclidean space and its dimensions are discussed. Subsequently, the properties and applications of objects such as open and closed sets, bounded and unbounded sets in R^m space, are examined [5].

This topic not only highlights theoretical issues but also helps students and professionals interested in mathematics gain a practical understanding of Euclidean space.

RESEARCH METHOD

The research conducted in this study is theoretical and analytical, focusing on the fundamental properties and applications of Euclidean space R^m [6]. The methodology is divided into the following steps:

1. Defining Euclidean Space R^m

The study begins by formalizing the mathematical definition of R^m, which comprises all ordered systems $(x_1,x_2,\dots,x_m)$ of mm real numbers. The concept of distance between two points in R^m is defined using the Euclidean metric (1). The metric properties such as non-negativity, symmetry, and the triangle inequality are rigorously proven [7].

2. Analysis of Sequences in R^m

To explore the behavior of sequences, the study defines convergence and boundedness in R^mthrough formal criteria. Proofs are provided to demonstrate the uniqueness of sequence limits (Lemma 2) and the relationship between limit existence and sequence boundedness (Lemma 1).

3. Investigation of Important Sets

Key subsets of R^m, including open balls, closed balls, mm-dimensional spheres, parallelepipeds, and simplices, are analyzed in detail. Formal definitions and mathematical expressions are presented to characterize these sets, with specific emphasis on their geometric and analytical significance.

4. Proof-Based Approach

Mathematical proofs are central to this research. The study employs inequalities such as the Cauchy–Bunyakovsky inequality to derive critical results, including the triangle inequality. Each lemma and theorem is accompanied by rigorous proofs to ensure mathematical accuracy and validity.

5. Application and Generalization

The theoretical results are linked to practical applications, highlighting their importance in mathematical analysis, geometry, and related fields. The study emphasizes the role of these properties in solving theoretical problems and developing analytical tools for applied mathematics.

6. Logical Framework

A logical progression is maintained throughout the analysis, beginning with foundational definitions and gradually building to more complex concepts. The study employs a formal mathematical structure to ensure clarity and coherence.

RESULTS AND DISCUSSION

1. m-Dimensional Euclidean Space

Consider the set consisting of all possible ordered systems $(x_1, x_2, \dots, x_m)$ of m real numbers $x_1, x_2, \dots, x_m$. Each ordered system $(x_1, x_2, \dots, x_m)$ represents an element of this set and is typically denoted by letters from the Latin alphabet. For example, $M(x_1, x_2, \dots, x_m)$, $A(x_1, x_2, \dots, x_m)$ etc., or equivalently as $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_m)$, $z = (z_1, z_2, \dots, z_m)$.

In the notation $M(x_1, x_2, \dots, x_m)$ the point M is said to have coordinates $x_1, x_2, \dots, x_m$ where x_1 is the first coordinate, x_2 is the second coordinate, and so on, up tox_m , which is the m-th coordinate. If we define the distance between two points $A(x_1, x_2, \dots, x_m)$ and $B(y_1, y_2, \dots, y_m)$ in this set by the formula:

$$\rho(A, B) = \sqrt{\sum_{i=1}^{m} (x_1 - y_1)^2} \qquad (1)$$

Then this set is called the m-dimensional Euclidean space or the Euclidean space R^m.

The introduced concept of distance has the following properties:

a. $\rho(A,B) \geq 0$ va $\rho(A,B) = 0 \ A = B$.

b. $\rho(A,B) = \rho(B,A)$.

c. 3)$\rho(A,C) \leq \rho(A,B) + \rho(B,C)$.

From property (1), we see that $\rho(A,B)$ is always non-negative.

If $\rho(A,B) = 0$ then $x_1 - y_1 = 0, x_2 - y_2 = 0, \ldots, x_m - y_m = 0$ which implies, $x_1 = y_1$, $x_2 = y_2, \ldots, x_m = y_m$ meaning $A = B$. Conversely, if $A = B$ then $x_1 = y_1$, $x_2 = y_2, \ldots$, $x_m = y_m$. From this, it follows that $\rho(A,B) = 0$. Thus, property (1) holds true. For property (2), the proof follows directly from the equality $(x_1 - y_1)^2 = (y_1 - x_1)^2$, $i = \overline{1,m}$ which is derived from (1). To prove property (3), we first establish the inequality known as the **Cauchy–Bunyakovsky inequality**:

$$\sqrt{\sum_{i=1}^{m}(a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{m} a_i^2} + \sqrt{\sum_{i=1}^{m} b_i^2} \quad (2)$$

It is known that for $\forall x \in R$, the following inequality holds:

$$\sum_{i=1}^{m}(a_i x + b_i)^2 \geq 0$$

Where x is a variable, and a_i, b_i are known constants.

From this inequality, we obtain the relation:

$$\left(\sum_{i=1}^{m} a_1^2\right)x^2 + 2\left(\sum_{i=1}^{m} a_1 b_1\right)x + \sum_{i=1}^{m} b_i^2 \geq 0$$

The expression on the left-hand side is a quadratic trinomial in terms of xxx. Since this quadratic trinomial cannot be negative, its discriminant must satisfy the inequality:

$$-\sum_{i=1}^{m} a_1^2 \sum_{i=1}^{m} b_i^2 + (\sum_{i=1}^{m} a_1 b_1)^2 \leq 0$$

From this, we obtain the inequality:

$$\sum_{i=1}^{m} a_1 b_1 \leq \sqrt{\sum_{i=1}^{m} a_i^2} \cdot \sqrt{\sum_{i=1}^{m} b_i^2}$$

Using this inequality, we can show that:

$$\sqrt{\sum_{i=1}^{m}(a_i + b_i)^2} = \sqrt{\sum_{i=1}^{m} a_i^2 + 2\sum_{i=1}^{m} a_1 b_1 + \sum_{i=1}^{m} b_i^2} \leq$$

$$\leq \sqrt{\left(\sum_{i=1}^{m} a_i^2\right)^2 + 2\sqrt{\sum_{i=1}^{m} a_i^2} \cdot \sqrt{\sum_{i=1}^{m} b_i^2} + \left(\sum_{i=1}^{m} b_i^2\right)^2} =$$

$$= \sqrt{\sum_{i=1}^{m} {a_i}^2} + \sqrt{\sum_{i=1}^{m} {b_i}^2}$$

This shows that inequality (2) holds.

Now, in inequality (2) of the Cauchy–Bunyakovsky inequality, if we let $a_i = x_i - y_i$, $b_i = y_i - z_i$ we get $a_i + b_i = x_i - z_i$. Thus, we obtain the inequality:

$$\sqrt{\sum_{i=1}^{m}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{m}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{m}(y_i - z_i)^2}$$

His proves property (3). This inequality, expressed in terms of property (3), is commonly referred to as the **triangle inequality** (which states that the length of one side of a triangle is not greater than the sum of the lengths of the other two sides).

2. Sequence in the R^m space. Important sets in the R^m space

Let the set of natural numbers N and the R^m space be given. For each $n (n \in N)$, a specific point in R^m is assigned as

$$x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}\right) \in R^m \ i.e.,$$
$$1 \to x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, \dots, x_m^{(1)}\right),$$
$$2 \to x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, \dots, x_m^{(1)}\right),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$n \to x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}\right),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Then, the collection of ordered points $x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots$ is called a sequence of numbers in R^m and is denoted as $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}\right)$.

Given a sequence of points in R^m

$$x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots \qquad (3)$$

and a point $a = (a_1, a_2, \dots, a_m) \in R^m$, let it be given.

Definition 1. If for every $\forall \varepsilon > 0$ there exists an $n_0 \in N$ such that for all $n > n_0$

$$\rho\left(x^{(n)}, a\right) < \varepsilon (4)$$

If the inequality is satisfied, the point a is called the limit of the sequence $x^{(n)}$, and it is denoted as $\lim_{n \to \infty} x^{(n)} = a$ or $n \to \infty$ da $x^{(n)} \to a$.

If the sequence (3) has a limit, it is called a convergent sequence. If there is no aaa that satisfies the condition in the definition of the limit, then the sequence $x^{(n)}$ is said to have no limit, and the sequence itself is called divergent. In the R^m space, a sequence can also be denoted as $\{x_n\} = \{x_{n1}, x_{n2}, \dots, x_{nm}\}$, and the definition 1 can be rewritten as follows.

Definition 2. Let $\{x_n\} = \{x_{n1}, x_{n2}, \dots, x_{nm}\}$ be a sequence in R^m. If there exists an element $a = (a_1, a_2, \dots, a_m)$ such that,

$$\lim_{n\to\infty} \rho(x_n, a) = 0$$

Then the point a is called the limit of the sequence x_n, and it is written as $\lim_{n\to\infty} x_n = a$. f there exists a constant $C \in R$ and an element $a \in R^n$ such that for all $\forall n \in N$, the inequality $\rho(x_n, a) \le C$ is satisfied, then the sequence x_n is called bounded.

Lemma 1. If the sequence x_n has a limit, then this sequence is bounded.

Proof. Let $\lim_{n\to\infty} x_n = a$. Then, by the definition of the limit, we have: $\lim_{n\to\infty} \rho(x_n, a) = 0$.

From this, it follows that the sequence $\rho(x_n, a)$ is a real-valued sequence that converges to 0. Therefore, there exists a positive constant C such that for all $n \in N$, the inequality

$$\rho(x_n, a) \le C$$

Holds, this shows that the sequence x_n is bounded.

Lemma 2. A sequence x_n cannot have two distinct limits.

Proof. Let $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$. According to the triangle inequality, for any $n \in N$, the following inequality holds:

$$0 \le \rho(a, b) \le \rho(a, x_n) + \rho(x_n, b)$$

Since $\rho(a, x_n)$ and $\rho(x_n, b)$ are sequences that tend to 0 as $n \to \infty$, we conclude that $\rho(a, b) = 0$. Therefore, we have $a = b$. This shows that the sequence cannot have two distinct limits.

Definition 3. In R^m space, the set of all points $x = (x_1, x_2, \dots, x_m)$ that satisfy the inequality $\rho(x, a) < r$ is called the open ball centered at a with radius r. This ball is denoted as $S_r(a)$ i.e.,

$$S_r(a) = x = (x_1, x_2, \dots, x_m) \in Rm \colon \rho(x, a) < r =$$
$$= \{x \in Rm \colon (x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_m - a_m)^2 < r^2\}.$$

Similarly, we can define the closed ball $S_r(a)$ as the set of points that satisfy the inequality:

$$\overline{S_r}(a) = x \in Rm \colon \rho(x, a) \le r =$$
$$= \{x \in R^m \colon (x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_m - a_m)^2 = r^2\}.$$

The set of points where is $\rho(x, a) = r$ called the m-dimensional sphere, and it is given by:

$$x = (x_1, x_2, \dots, x_m) \in Rm \colon \rho(x, a) = r =$$
$$= \{x \in Rm \colon (x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_m - a_m)^2 = r^2\}.$$

Definition 4. The sets

$$\{x \in R^m \colon a_1 < x_1 < b_1, a_2 < x_2 < b_2, \dots, a_m < x_m < b_m\},$$
$$\{x \in R^m \colon a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \dots, a_m \le x_m \le b_m\}$$

(Where $a_i, b_i \in R, i = \overline{1, m}$) are called an m −dimensional parallelepiped and a closed parallelepiped**,** respectively.

Definition 5. The set

$\{x \in R^m \colon x_1 \ge 0, x_2 \ge 0, \dots, x_m \ge 0,\ x_1 + x_2 + \dots + x_m \le h\}$ (where $h > 0$) is called an m-dimensional simplex**.**

Task 1. For the convergence of the sequence $\{x_n\}$ in R^m to the point a, it is necessary and sufficient that for any open ball $S_r(a)$ all elements of the sequence, except the limit elements, must remain within that ball.

Lemma 3. For the sequence $\{x_n\} = \{x_{n1}, x_{n2}, \dots, x_{nm}\}$ in R^m to converge to the point $a = (a_1, a_2, \dots, a_m)$ it is necessary and sufficient that the following equations hold:

$$\lim_{n\to\infty} x_n = a_i, i = \overline{1,m}.$$

Proof. Let $\lim_{n\to\infty} x_n = a$. Then, $\lim_{n\to\infty} \rho(x_n, a) = 0$. Therefore, for any $i = 1,2,\dots,m$ the following inequality holds:

$$0 \le |x_{ni} - a_i| \le \left(\sum_{k=1}^{m} (x_{nk} - a_k)^2\right)^{\frac{1}{2}} = \rho(x_n, a)$$

As $n \to \infty$, *the expression* $|x_{ni} - a_i|$ tends to 0. This implies that $\lim_{n\to\infty} x_n = a_i$. Conversely, if for any $i = 1,2,\dots,m$ uchun $\lim_{n\to\infty} |x_{ni} - a_i| = 0$ then as $n \to \infty$ da

$$\rho(x_n, a) = \left(\sum_{k=1}^{m} (x_{ni} - a_i)^2\right)^{\frac{1}{2}} \to 0.$$

From this, it follows that $\lim_{n\to\infty} x_n = a$.

If for all $\forall \varepsilon > 0$, *there exists* $n_0 \in N$ *such that for all* $\forall n > n_0$ *and gor all* $\forall k > n_0$ the relation $\rho(x_n, x_k) < \varepsilon$ holds, then the sequence $\{x_n\}$ k is called a **fundamental sequence** in R^m.

CONCLUSION

Fundamental Finding : This article clarified the essential concepts of R^m space, emphasizing sequences, open and closed sets, and bounded and unbounded sets, which are fundamental to mathematical analysis and geometry. Understanding these concepts provides a solid theoretical foundation for solving both theoretical and practical problems. **Implication :** The insights into R^m space and its properties facilitate a deeper understanding of mathematical models, enabling their application to complex problem-solving in areas such as mathematical analysis and linear algebra. This knowledge is pivotal for advancing research and practical applications. **Limitation :** While the article covers the basics of R^m space, it is primarily theoretical and lacks detailed case studies or applied examples, which may limit its accessibility for practitioners who rely on real-world applications to fully grasp the concepts. **Future Research :** Future studies could expand on this work by exploring more applied aspects of R^m space, such as numerical methods or computational techniques, to enhance its practicality and demonstrate its relevance in solving real-world mathematical challenges.

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